A Equivariant Hochschild homology

The aim of this appendix is to introduce the equivariant Hochschild homology of an algebra acted by an algebraic group. For a commutative algebra this is an explicit form of the derived loop stack introduced in [TV]. In the body of the paper we only need the Bott morphism (Definition A.2) and its key property given by Proposition A.3. The general definition is given to motivate it. The proof of the proposition is based on general facts about cyclic and simplicial objects. They are collected in the first three subsections. For a more careful exposition we refer to [Lod] and references therein.

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A.1 Simplicial objects

Let $\mathbf{\Delta}$ denote the simplicial category, with objects non-empty linear orders $[n] = \{0 < 1 < ... < n\}$ for $n \in \mathbb{N}$ and morphisms $Hom_{\mathbf{\Delta}}([m],[n])$ all order-preserving maps from $[m]$ to $[n]$. Morphisms in $\mathbf{\Delta}$ are generated by faces $\delta_i\colon [n-1] \to [n]$ for which every element except the ith one has exactly one preimage, and degeneracies $\sigma_i\colon [n+1] \to [n]$ for which every element except the ith one has exactly one preimage. The corresponding morphisms in $\mathbf{\Delta}^{op}$ are denoted by d_i and s_i. The simplex category may be realized as a full subcategory of the category of small categories consisting of path categories of finite linear quivers

$$q^n\colon \mathbf{0} \longrightarrow \mathbf{1} \longrightarrow \ldots \longrightarrow \mathbf{n}.$$

If $\mathcal{C}$ is a category, then a simplicial object F in $\mathcal{C}$ is a functor $F\colon \mathbf{\Delta}^{op} \to \mathcal{C}$, and a cosimplicial object G in $\mathcal{C}$ is a functor $G\colon \mathbf{\Delta} \to \mathcal{C}$. Denote the category of simplicial (respectively, cosimplicial) sets by **SSet** (respectively, **cSSet**). It is convenient to present a simplicial (respectively, cosimplicial) object F (respectively, G) as a sequence of objects F_i (respectively, G^i) and morphisms δ_i, σ_i (respectively, d_i, s_i) between them. Sets $Hom_{\mathbf{\Delta}}([m],[n])$ form a cosimplicial simplicial set. The simplicial set $Hom_{\mathbf{\Delta}}(-,[n])$ is denoted by Δ^n and is called (the standard) n-simplex.

The product on the category of (co)simplicial sets is given by the diagonal map $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$ in combination with product of sets. For a pair of simplicial sets the function complex $\mathbf{Hom}(X,Y)$, which is a simplicial set, is defined by the formula

$$\mathbf{Hom}(X,Y) = Hom_{\mathbf{\Delta}}(X \times \Delta^\bullet, Y),$$

where $Hom_{\mathbf{\Delta}}$ means morphisms in the category of simplicial sets, the simplicial structure on $\mathbf{Hom}(X,Y)$ is induced by the cosimplicial structure on $\Delta^\bullet$. For a pair of cosimplicial sets the definition is the same. Functors $\times$ and **Hom** are connected by the exponential law:

$$Hom_{\mathbf{\Delta}}(Z, \mathbf{Hom}(X,Y)) = Hom_{\mathbf{\Delta}}(X \times Z, Y),$$

that is $X \times -$ is left adjoint to $\mathbf{Hom}(X,-)$.

Let $K^\bullet$ be a cosimplicial simplicial set. Define the simplicial set $\operatorname{Tot} K^\bullet$ by

$$\operatorname{Tot} K^\bullet = \mathbf{Hom}_{\mathbf{\Delta}^{op}}(\Delta^\bullet, K^\bullet),$$

where subscript $\mathbf{\Delta}^{op}$ means that we consider only maps that commute with the cosimplicial structures on both sides (see e. g. [GJ, Ch. VII.5]).

The functor $- \times \Delta^\bullet$ from simplicial sets to cosimplicial simplicial sets is left adjoint to Tot.

Fix an additive category. We shall name its objects by modules. For a simplicial module E, that is a simplicial object in the additive category, the associated chain complex $Ch(E)$ in this category is defined as follows. Let $\Delta_n = Hom_{\mathbf{\Delta}}([n], -)$ be the standard cosimplicial set. Let D_{-n} be the free $\mathbb{Z}$-module generated by Δ_n. Sequence $D_\bullet$ is a simplicial objects in cosimplicial $\mathbb{Z}$-modules. Define the morphism of cosimplicial $\mathbb{Z}$-modules $b\colon D_{-n} \to D_{-n+1}$ by $b = \sum_i (-1)^i \delta_i$, where δ_i are face maps. This is a differential. Denote the corresponding complex of cosimplicial $\mathbb{Z}$-modules by D. Then for a simplicial module E the complex $Ch(E)$ is given by

$$Ch(E) = D \otimes_{\mathbf{\Delta}} E,$$

where $\otimes_{\mathbf{\Delta}}$ means the sum $\coprod D^n \otimes E_n / \approx$, where the equivalence relation $\approx$ is generated by $x \otimes \phi_*(y) \approx \phi^*(x) \otimes y$ for any $x \in D^m$, $y \in E_n$ and $\phi \in Hom_{\mathbf{\Delta}}([n], [m])$. Complex D is a resolution of the trivial $\mathbb{Z}$-module in the category of cosimplicial $\mathbb{Z}$-modules, thus functor Ch is the derived tensor product with the trivial $\mathbb{Z}$-module.

In the same way for a cosimplicial module E the cochain complex $cCh(E)$ is defined.

A cosimplicial simplicial module E gives the bicomplex $cCh(Ch(E))$. There is a natural morphism from the chain complex $Ch(\operatorname{Tot} E)$ to the total complex $\operatorname{Tot} cCh(Ch(E))$ of this bicomplex.

A.2 Cyclic objects

Let $\mathbf{\Lambda}$ denote the cyclic category, with objects non-empty linear orders with a free action of $\mathbb{Z}$ and morphisms all order-preserving maps of orders that commutes with the action of $\mathbb{Z}$ taken up to this action. Objects of $\mathbf{\Lambda}$ are numerated by non-negative integers, object $[n]$ is presented by the set $\mathbb{Z}$ with the standard order and the generator of $\mathbb{Z}$ acts on it by shift by $n+1$. Analogously with the simplex category morphisms in $\mathbf{\Lambda}$ are generated by faces δ_i, degeneracies σ_i and in addition by cyclic operators $\tau_n \in Aut([n]) = \mathbb{Z}_{n+1}$ that is a generator of the cyclic group. Category $\mathbf{\Lambda}$ contains $\mathbf{\Delta}$ as a subcategory, morphisms of the latter are generated by faces and degeneracies excluding the extra one $\sigma_{n+1} \in Hom_{\mathbf{\Lambda}}([n+1], [n])$. Denote this inclusion and the opposite functor by $j\colon \mathbf{\Delta}^{op} \hookrightarrow \mathbf{\Lambda}^{op}$ and $j^{op}\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Lambda}$. The cyclic category may be realized as a subcategory of the category of small categories consisting of path categories of cyclic quivers

$$c^n\colon\ \mathbf{0} \longrightarrow \mathbf{1} \longrightarrow \ldots \longrightarrow \mathbf{n} \quad (\text{with an arrow } \mathbf{n} \to \mathbf{0})$$

and functors between them that induce maps of degree 1 on nerves.

If $\mathcal{C}$ is a category, then a cyclic object F in $\mathcal{C}$ is a functor $F\colon \mathbf{\Lambda}^{op} \to \mathcal{C}$, and a cocyclic object G in $\mathcal{C}$ is a functor $G\colon \mathbf{\Lambda} \to \mathcal{C}$. Denote the category of (co)simplicial sets by **CSet** (**cCSet**). Sets $Hom_{\mathbf{\Lambda}}([m],[n])$ form a cocyclic cyclic set. The cyclic set $Hom_{\mathbf{\Lambda}}(-,[n])$ is denoted by Λ^n and called the standard cyclic set.

There are forgetful functors $j_*\colon \mathbf{CSet} \to \mathbf{SSet}$ and $j_*^{op}\colon \mathbf{cCSet} \to \mathbf{cSSet}$ induced by embeddings $j\colon \mathbf{\Delta}^{op} \hookrightarrow \mathbf{\Lambda}^{op}$ and $j^{op}\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Lambda}$. The result of the forgetful functor is called the (co)simplicial set associated with a (co)cyclic set.

The product on **CSet** (**cCSet**) is defined in the same way as for (co)simplicial sets. For a pair of cyclic sets the function complex $\mathbf{Hom}(X,Y)$, which is a cyclic set, is defined by the formula

$$\mathbf{Hom}(X,Y) = Hom_{\mathbf{\Lambda}}(X \times \Lambda^{\bullet}, Y),$$

where $Hom_{\mathbf{\Lambda}}$ means morphisms in the category of cyclic sets, the cyclic structure on the result comes from the cocyclic structure on $\Lambda^{\bullet}$. The function complex commutes with the forgetful functor j_*: there is an isomorphism

$$j_*\,\mathbf{Hom}(X,Y) = \mathbf{Hom}(j_*X, j_*Y)$$

for cyclic X and Y.

The functor $j_*\colon \mathbf{CSet} \to \mathbf{SSet}$ has a left adjoint $j^*\colon \mathbf{SSet} \to \mathbf{CSet}$. Their composition acts as

$$j_*j^*X = \mathrm{C} \times X,$$

where C is the simplicial set associated with cyclic set $j^*\mathrm{pt}$. This cyclic set is defined by $(j^*\mathrm{pt})_n = \mathbb{Z}_{n+1}$, $Aut([n])$ acts freely on $(j^*\mathrm{pt})_n$. Simplicial set C has only two non-degenerate simplexes in degree 0 and 1 and may be thought as a simplicial model of circle. The functor j_*j^* is a monad, in particular there is the canonical map $\mathrm{C}\times\mathrm{C} = j_*j^*j_*j^*\mathrm{pt} \overset{can}{\to} j_*j^*\mathrm{pt} = \mathrm{C}$ that presents the group structure on circle. Objects in the image of $j_*\colon \mathbf{CSet} \to \mathbf{SSet}$ are modules over this monad:

$$j_*j^*j_*X \overset{can}{\to} j_*X.$$

Thus cyclic sets may be thought as simplicial sets with circle action.

Let $K^{\bullet}$ be a cocyclic simplicial set. Define the cyclic set $\mathrm{Tot}\,K^{\bullet}$ by

$$\mathrm{Tot}\,K^{\bullet} = \mathbf{Hom}_{\mathbf{\Lambda}^{op}}(\Lambda^{\bullet}, j^*K^{\bullet}),$$

where **Hom** is the function complex in the category of cyclic sets and subscript $_{\mathbf{\Lambda}^{op}}$ means that we consider only maps that commute with the cocyclic structures on both sides. One may check that $j_*\,\mathrm{Tot}\,K^{\bullet} = \mathrm{Tot}\,j_*^{op}K^{\bullet}$, where j_*^{op} is the forgetful functor from cocyclic simplicial to cosimplicial simplicial sets.

Functor Tot has left adjoint from cyclic to cocyclic simplicial sets defined by $X \mapsto X \times_{\mathrm{C}} \Lambda^{\bullet}$, where $\times_{\mathrm{C}}$ means the product of cyclic sets with nth component factorized by $\mathbb{Z}_{n+1}$. The cocyclic structure comes from the second factor and the simplicial structure is given by the natural isomorphism of sets $X \times_{\mathrm{C}} \Lambda^n = j_*X \times \Delta^n$.

Fix an additive category. Let E be a cyclic module, that is a cyclic object in this additive category. The associated simplicial module j_*E is a module over the monad $- \otimes \mathbb{Z}[\mathrm{C}]$, where $\mathbb{Z}[\mathrm{C}]$ is the free $\mathbb{Z}$-module generated by simplicial set C. This follows that the chain complex $Ch(j_*E)$ is acted by $Ch(\mathbb{Z}[\mathrm{C}])$, homology of the last complex is homology of circle.

The cyclic chain complex $CCh(E)$ plays the role of the equivariant chains with respect to the circle action and may be defined in three equivalent ways:

1. Simplicial object j_*E is a module over the monad j_*j^*. The cyclic chain complex $CCh(E)$ is the homology of this monad with coefficients in this module, that is the total complex of the bisimplicial module

$$\underbrace{j_*j^* \cdots j_*j^*}_{n} j_*E = \underbrace{\mathbb{Z}[\mathrm{C}] \otimes \cdots \otimes \mathbb{Z}[\mathrm{C}]}_{n} \otimes j_*E$$

   with the standard simplicial structure.

2. Complex $CCh(E)$ presents the derived tensor product with the trivial module in the category of cocyclic modules.

3. $CCh(E) = (Ch(j_*E) \otimes \mathbb{Z}[u], b + uB)$, where u is of degree 2, b is the usual differential on $Ch(j_*E)$ and B is a differential of degree -1, for a precise formula for it see [Lod, Ch. 2.1].

Equivalence of first two definitions is a standard fact about triple homology, the second and the third definitions are equivalent by [Lod, Ch. 6.2]. Operator B presents action of 1-cycle in $Ch(\mathbb{Z}[\mathrm{C}])$.

From a cocyclic simplicial module E functor Tot produces a cyclic module. This action respects operator B up to homotopy by the following proposition.

Proposition A.1. *For a cocyclic simplicial module E the natural morphism of complexes $Ch(j_* \operatorname{Tot} E) \to \operatorname{Tot} cCh(j_*^{op} Ch(E))$ commutes with operators B up to a boundary.*

Proof. Let $T^{\bullet}_{\bullet,\bullet}$ be the cyclic cyclic cosimplicial set with $T^m_{n,\bullet} = \operatorname{Tot} j_*\Delta^m \times \Lambda_n$. Then $\operatorname{Tot} E = \mathbb{Z}[T^{\bullet}_{\bullet,\bullet}] \otimes_{\mathbf{\Delta} \times \mathbf{\Lambda}^{op}} E$ for a cocyclic simplicial complex E, where the tensor product has the same sense as in the end of the previous subsection. This follows that it is enough to prove the statement only for free cocyclic simplicial $\mathbb{Z}$-modules. Then the statement follows from the same one for the functor adjoint to Tot on free $\mathbb{Z}$-modules. The last statement may be proved by the acyclic models method, for details see [Jon, Th. 4.1]. □

A.3 Cyclic nerve

Let $\mathcal{C}$ be a category. The nerve of $\mathcal{C}$ is the simplicial set $N\mathcal{C}_\bullet = \operatorname{Fun}(q^\bullet, \mathcal{C})$, the simplicial structure comes from the cosimplicial structure on $q^\bullet$. That is

$$N_n\mathcal{C} = \{ A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n \mid A_i \in \operatorname{Ob}\mathcal{C}, f_i \in \operatorname{Mor}\mathcal{C}\},$$

face maps are given by composition of morphisms, degeneracies — by insertion of identity.

The cyclic nerve of $\mathcal{C}$ is the cyclic set $CN\mathcal{C}_\bullet = \mathrm{Fun}(c^\bullet, \mathcal{C})$, the cyclic structure comes from the cocyclic structure on $c^\bullet$. That is

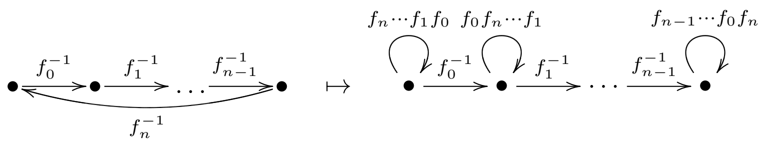
$$CN_n\mathcal{C} = \{\, A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n,\ A_n \xrightarrow{f_n} A_0 \mid A_i \in \mathrm{Ob}\,\mathcal{C}, f_i \in \mathrm{Mor}\,\mathcal{C}\},$$

face maps are given by composition of morphisms, degeneracies — by insertion of identity and cyclic operator — by rotations of loop of morphisms.

Let $\mathcal{G}$ be a groupoid. The inertia groupoid $I\mathcal{G}$ is the category with

$$\mathrm{Ob}\, I\mathcal{G} = \{(A, a) \mid A \in \mathrm{Ob}\,\mathcal{G}, a \in Aut\, A\}$$
$$Hom_{I\mathcal{G}}((A, a), (B, b)) = \{x : ax = xb \mid x \in Hom_{\mathcal{G}}(A, B)\}.$$

The simplicial set associated with the cyclic nerve $j_* CN_\bullet\mathcal{G}$ is isomorphic to the nerve of the inertia groupoid $N_\bullet I\mathcal{G}$. The isomorphism is given by



The inertia groupoid is the category of functors $\mathrm{Fun}(\Sigma\mathbb{Z}, \mathcal{G})$, where $\Sigma\mathbb{Z}$ is the category with one object and the automorphism group $\mathbb{Z}$. Thus it is acted by $\Sigma\mathbb{Z}$ and the nerve is acted by $N_\bullet\Sigma\mathbb{Z}$. The simplicial set $N_\bullet\Sigma\mathbb{Z}$ is another than C simplicial model of circle. Its action on $N_\bullet I\mathcal{G}$ is equivalent to the action of C on $j_* CN_\bullet\mathcal{G}$ under the isomorphism above ([Lod, Prop. 7.3.4]).

The embedding $\mathcal{G} \to I\mathcal{G}$ sending A to $(A, 1)$ induces the map of nerves $N_\bullet\mathcal{G} \to j_* CN_\bullet\mathcal{G}$: a sequence of arrows goes to a cycle of arrows with the identical product. This map induces an embedding of cyclic sets

$$j^* N_\bullet\mathcal{G} \to j^* j_* CN_\bullet\mathcal{G}.$$

The last cyclic set may be realized as a set of cycles of arrows in $\mathcal{G}$ with a marked arrow.

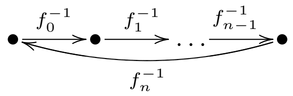
Consider generators, that is the set of quiver arrows of the category c^i. This is a cyclically ordered set. A functor between c^i and c^j that is of degree 1 on nerves induces a morphism between this cyclically ordered sets in the opposite direction: an arrow goes to its preimage. Conversely, any morphism between cyclically ordered sets of arrows uniquely defines such a functor. This gives a functor from category $\mathbf{\Lambda}$ to the category of small categories, that is a cyclic object in categories. Denote it by $c_\bullet$. Alternatively one may say that category $\mathbf{\Lambda}$ is isomorphic to its opposite and the cyclic object $c_\bullet$ is induced from the cocyclic object $c^\bullet$ under this isomorphism.

For a category $\mathcal{C}$ introduce a cocyclic simplicial object $N_\bullet\,\mathrm{Fun}(c_\bullet, \mathcal{C})$, the cocyclic structure comes from the cyclic structure on $c_\bullet$.

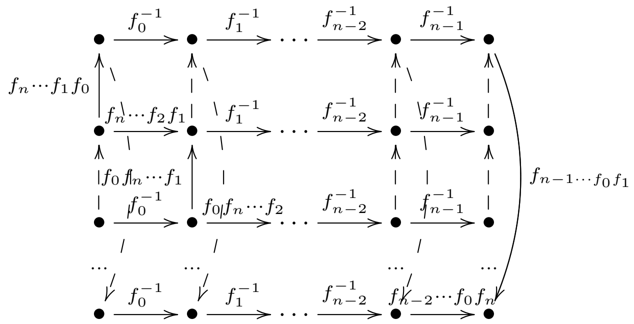
Proposition A.2. *For a groupoid $\mathcal{G}$ there is a natural isomorphism*

$$\mathrm{Tot}\, N_\bullet \operatorname{Fun}(c_\bullet, \mathcal{G}) = CN_\bullet \mathcal{G}.$$

Proof. Any element of $\mathrm{Tot}\, K^\bullet = \mathbf{Hom}_{\mathbf{\Lambda}^{op}}(\Lambda^\bullet, j^*K^\bullet)$ is defined by a nonnegative integer n and image of (any) non-degenerate n-simplex of Λ^n. Define an isomorphism $CN_\bullet\mathcal{G} \to \mathrm{Tot}\, N_\bullet \operatorname{Fun}(c_\bullet, \mathcal{G})$ by sending simplex



to the element of $j^*N_n \operatorname{Fun}(c_n, \mathcal{G})$ given by



Here dotted arrows are marked by the unit, columns presents elements of $\operatorname{Fun}(c_\bullet, \mathcal{G})$, rows presents j_*N. The latter cyclic set is realized as a subset of the cyclic nerve, the first arrow is marked, the closing arrow is not drawn.

Note that the restriction of this morphism on $CN_\bullet\mathcal{G} \to N_\bullet \operatorname{Fun}(c_0, \mathcal{G})$ is the described above isomorphism between the cyclic nerve and the nerve of inertia. $\square$

Everything in this subsection is applicable for algebraic groupoids, that is when morphism and objects are affine schemes. Then nerves and cyclic nerves are simplicial and cyclic schemes.

A.4 Equivariant Hochschild homology

In this subsection all modules are objects of an additive tensor category, say modules over a commutative ring.

Let module H be a commutative Hopf algebra with coproduct Δ, antipode S and counit ε. Denote by G the algebraic group represented by H. Representation V of G gives a comodule $\rho\colon V \to V \otimes H$ that obeys the associativity condition. The coproduct $\Delta\colon H \to H \otimes H$ equips H with a comodule structure.

This representation is called the left regular one. The right regular representation is isomorphic to the left by conjugation by S. Denote left and right actions by gh and hg, where $g \in G$, $h \in H$.

For a representation (V, ρ) introduce the cosimplicial module $C^\bullet(G, H)$ by

$$C^n(G, V) = V \otimes H^{\otimes n}$$

$$\delta^0 = \rho \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \quad \delta^i = \mathrm{id} \otimes \cdots \otimes \overset{i}{\Delta} \otimes \cdots \otimes \mathrm{id} \quad \delta^{n+1} = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes 1$$

$$\sigma^i = \mathrm{id} \otimes \cdots \otimes \overset{i}{\varepsilon} \otimes \cdots \otimes \mathrm{id}$$

Cohomology of the complex $cCh(C^\bullet(G, H))$ is called the cohomology of representation V.

For an algebraic group G introduce the cocyclic algebraic group G by

$$\mathsf{G}^n = G^{\times(n+1)}$$

$$\delta^i(g_0, \ldots, g_n) = \begin{cases} (g_0, \ldots, g_i, g_i, \ldots, g_n) & , i < n \\ (g_0, g_1, \ldots, g_0) & , i = n \end{cases}$$

$$\sigma^i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n)$$

$$\tau^n(g_0, g_1, \ldots, g_n) = (g_1, \ldots, g_n, g_0)$$

Introduce the cyclic module $O_\bullet$ that is a representation of G by

$$O_{n+1} = H^{\otimes(n+1)}$$

$$(g_0, \ldots, g_n) h_0 \otimes \cdots \otimes h_n = g_1 h_0 g_0^{-1} \otimes g_2 h_1 g_1^{-1} \otimes \cdots \otimes g_0 h_n g_n^{-1}$$

$$d_i(h_0 \otimes \cdots \otimes h_n) = h_0 \otimes \cdots \otimes \varepsilon(h_i) \otimes \cdots \otimes h_n$$

$$s_i(h_0 \otimes \cdots \otimes h_n) = \begin{cases} h_0 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n & , i < n \\ \Delta_2(h_n) \otimes h_0 \otimes \cdots \otimes \Delta_1(h_n) & , i = n \end{cases}$$

$$t_n(h_0 \otimes h_1 \otimes \cdots \otimes h_n) = h_n \otimes h_0 \otimes \cdots \otimes h_{n-1}$$

Let module A be a G-algebra, that is an associative algebra and representation of G such that the multiplication $A \otimes A \dot{\to} A$ is a morphism of representations. Define the cyclic module $A_\bullet$ that is a representation of G by

$$A_{n+1} = A^{\otimes(n+1)}$$

$$(g_0, \ldots, g_n)\, a_0 \otimes \cdots \otimes a_n = g_0 a_0 \otimes \cdots \otimes g_n a_n$$

$$d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_n & , i < n \\ a_n \cdot a_0 \otimes \cdots \otimes a_{n-1} & , i = n \end{cases}$$

$$s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes \overset{i}{1} \otimes \cdots \otimes a_n$$

$$t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

Definition A.1. *For an algebraic group G and a G-algebra A define the equivariant Hochschild chain complex $C_*^G(A)$ as the total complex of the bicomplex* *associated with the cyclic cosimplicial module* $C^\bullet(\mathsf{G}, (A \otimes O)_\bullet)$*:*

$$C_*^G(A) = \operatorname{Tot} Ch(j_*\, cCh(C^\bullet(\mathsf{G}, (A \otimes O)_\bullet))).$$

Homology of this complex $HH_*^G(A)$ *is called the equivariant Hochschild homology.*

The cyclic structure on the chain complex gives the operator B on the equivariant Hochschild homology. One may define as well the cyclic equivariant homology and so on in a parallel way with the usual Hochschild homology, which is a particular case of the equivariant one with $G = 1$.

Let H^{ad} be the adjoint representation of G. Then by the previous subsection $cCh(C^\bullet(G, H^{ad}))$ may be thought as the ring of functions on the cyclic nerve of G and thus it is equipped with a cocyclic structure. The following proposition states that cohomology of this complex is G-equivariant Hochschild homology of the trivial algebra.

Proposition A.3. *For an algebraic group* G *there is a natural morphism*

$$cCh(C^\bullet(G, H^{ad})) \to C_*^G(1)$$

that is a homotopy equivalence of complexes and commutes with operator B *up to a coboundary.*

Proof. Consider the natural morphism

$$cCh(j_* \operatorname{Tot} C^\bullet(\mathsf{G}, O_\bullet)) \to \operatorname{Tot} Ch(j_*\, cCh(C^\bullet(\mathsf{G}, O_\bullet))).$$

As all $C^\bullet(\mathsf{G}, O_n)$ are homotopy equivalent to each other and the cyclic structure morphisms are homotopy equivalences, this morphism is a homotopy equivalence. Then note that $C^\bullet(\mathsf{G}, O_\bullet)$ is the space of functions on $N_\bullet \operatorname{Fun}(c_\bullet, G)$ and by Proposition A.2 this homotopy equivalence is the sought for. Proposition A.1 follows that it respects operator B up to a coboundary. □

Let G be an algebraic group over a commutative ring and $\mathcal{O}(T)$ be a commutative algebra such that its spectrum T is a torsor over G. Then

$$cCh(C^\bullet(G, \mathcal{O}(T))) = \mathcal{O}(T/G)$$

and therefore $C_*^G(\mathcal{O}(T))$ is homotopy equivalent to the total complex of the cyclic cosimplicial module $\mathcal{O}(G/T)_\bullet \otimes C^\bullet(\mathsf{G}, O_\bullet)$. Sending O_i to the structure sheaf by $\varepsilon^{\otimes(i+1)}$ we get a morphism $C_*^G(\mathcal{O}(T)) \to Ch(j_*\mathcal{O}(G/T)_\bullet) = C_*^1(\mathcal{O}(T/G))$.

Consider the composition of the Isomorphism from Proposition A.3, map induced by the embedding of the unit and the morphism given by the factorization:

$$cCh(C^\bullet(G, H^{ad})) \to C_*^G(1) \to C_*^G(T) \to C_*^1(\mathcal{O}(T/G)).$$

Definition A.2. *For an algebraic group G and a torsor T over it the composite above*

$$cCh(C^\bullet(G, H^{ad})) \to C^1_*(\mathcal{O}(T/G))$$

that is a morphism of complexes commuting with operator B up to a coboundary is called the Bott morphism.

The name is due to paper [Bot] where an analogous morphism is constructed.

The Bott morphism has a transparent meaning in terms of [TV]: this is the inverse image of functions on the derived loop space under the map from the stack with objects T/G and trivial morphism to the stack of G-torsors.

References

[Bot] R. Bott. On the Chern-Weil homomorphism and the continuous cohomology of Lie-groups. *Advances in Math.*, 11:289–303, 1973.

[GJ] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.*

[Jon] John D. S. Jones. Cyclic homology and equivariant homology. *Invent. Math.*, 87(2):403–423, 1987.

[Lod] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998.

[TV] Bertrand Toën and Gabriele Vezzosi. Chern character, loop spaces and derived algebraic geometry. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 331–354. Springer, Berlin, 2009.